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Lower Semicontinuity of Solution Mappings for Parametric Set Optimization Problems

Shasha Cao *, Yangdong Xu

School of Science, Chongqing University of Posts and Telecommunications, Chongqing 400065, China.

* Corresponding Author

Abstract

In this paper, we focus on the continuity of the solution mappings of parametric set optimization problems with the Minkowski difference. Under some suitable assumptions, the lower semicontinuity concerned with a nonlinear scalarization function for sets is first presented. Then, the lower semicontinuity of the solution mappings of parametric set optimization problems is established by the lower semicontinuity of the nonlinear scalarization function.

Keywords

Set optimization problem, Nonlinear scalarization function, Lower semicontinuity.

1. Introduction

In recent years, the extension of vector-valued optimization problem to set-valued optimization problem has received an increasing attention due to their wide applications in many fields such as optimal control, differential inclusions, game theory, robust optimization, fuzzy optimization, welfare economics and mathematical finance; see, for instance, [1-3] and the references therein. In the literature, there are mainly two types of criteria of solution for set-valued optimization problems. The classical one is the vector criterion: a solution of a set-valued map is defined via the minimal element of the image set of the map with respect to the usual ordering relation of vector optimization. Although it is of mathematical interest, it does not seem natural whenever one needs to consider preferences over sets, since only one element does not necessarily imply that the whole image set is in a certain sense minimal with respect to all image sets. In order to overcome this drawback, the set criterion is proposed by Kuroiwa [4]. The set criterion is that a solution is defined via the minimal set of the collection of all image sets with respect to set order relations. Regarding this criterion, there are some other set order relations considered in set-valued optimization problems. We refer the reader to [5,6] and the references therein for more details. A set-valued optimization problem with this criterion is called a set optimization problem.

It is well known that nonlinear scalarization functions are the most essential tools in vector or set optimization. There are mainly two types of scalarization functions in vector optimization. They are the Gerstewitz's function [1,7,8] and the oriented distance function [9,10]. Accordingly, in set optimization, there are also two kinds of scalarization functions have been used: the extensions of the Gerstewitz's function [6,11-14] and the extensions of the oriented distance function of Hiriart-Urruty [11,15]. In terms of scalarization techniques, several theoretical aspects of set optimization were discussed, such as characterization of several types of optimal solutions, alternative theorems, optimality conditions and the well-posedness; see, for instance, [11-18] and the references therein.

As we know, the continuity of the extensions of the Gerstewitz's function play a significant role in the research of the existence and the stability of set optimization problems. For more details, we refer the reader to [12,19]. Therefore, it is necessary and interesting to investigate the continuity for the extensions of the Gerstewitz's function. In this paper, we give the lower continuity of the nonlinear scalarization function proposed by Karaman et al. in [6], which is defined by a partial set order relation involving the Minkowski difference. As an application, we study the stability of set optimization problems.

The rest of the paper is organized as follows. In section 2, we introduce some definitions and previous results required throughout the paper. In Section 3, by using some properties of nonlinear scalarization functions defined in [6] and well-known results for set-valued mappings, we first show the lower semi continuity and convexity of a nonlinear scalarization function for sets. Then we explore an application of properties of the nonlinear scalarization function to the lowercontinuity of solution mappings of a parametric set optimization problem. In Section 4, we give the concluding remarks of the paper.

2. Preliminaries

In this section, we recall some basic definitions and properties which are necessary for this study. Throughout this paper, X and Y are two normed vector spaces. Given a subset A of Y, the closure, the complement, the topological interior, the boundary and the convex hull of A are denoted, respectively, by cl A, A^c, int A, bd A and conv A. We denote by $\mathbb{B}_Y$ the closed unit ball in Y, i.e., $\mathbb{B}_Y := \{a \in Y : \| a \| \leq 1\}$. The family of nonempty proper subsets of Y, the family of nonempty bounded subsets of Y and the family of nonempty compact subsets of Y are denoted by $\mathcal{P}_0(Y), \mathcal{B}^*(Y)$ and $\mathcal{B}^{**}(Y)$, respectively. For every $A, B \in \mathcal{P}_0(Y)$ and $\lambda \in \mathbb{R}$, we denote respectively

$$A - B = \{y_1 - y_2 : y_1 \in A, y_2 \in B\}, \ \ \lambda A = \{\lambda y : y \in A\}$$

by the algebraic difference of the sets A and B, and the scalar multiplication of the set A.

A nonempty subset $C \subseteq \mathbb{R}^m$ is said to be a cone if $tC \subseteq C$ for all $t \geq 0$. A cone $C \subseteq \mathbb{R}^m$ is said to be convex (resp. pointed) if and only if $C + C \subseteq C$ (resp. $C \cap (-C) = \{0_{\mathbb{R}^m}\}$). Throughout the paper, we assume that C is a closed, convex and pointed cone with nonempty interior. Let Y^* be the topological dual space of Y and the dual cone of C be denoted by C^*, which is defined by $C^* = \{f \in Y^* : f(c) \geq 0, \forall c \in C\}$.

For $A, B \in \mathcal{P}_0(Y)$, the Minkowski difference of A and B, considered in [20], is given by

$$B \dot{-} A := \{z \in Y : z + A \subseteq B\} = \bigcap_{a \in A} (B - a).$$

It is worth mentioning that the Minkowski difference of a set and a vector coincide with the algebraic difference of them, that is, $A \dot{-} b = A - b$ for all $A \in \mathcal{P}_0(Y)$ and $b \in Y$. In addition, $(A \dot{-} B) - b = A \dot{-} B - b = A \dot{-} (B + b)$ and $(A - b) \dot{-} B = (A \dot{-} B) - b$, for all $A, B \in \mathcal{P}_0(Y)$ and $b \in Y$.

We now recall some order relations on $\mathcal{P}_0(Y)$. The first one is the lower set order relation $\leq_C^l$ and the upper set order relation $\leq_C^u$ on $\mathcal{P}_0(Y)$, which are discussed by [4].

$$A \leq_C^l B \Leftrightarrow B \subseteq A + C; A \leq_C^u B \Leftrightarrow A \subseteq B - C.$$

What is noteworthy is that $\leq_C^l$ and $\leq_C^u$ are pre-order relations, i.e., reflexive and transitive relations, on $\mathcal{P}_0(Y)$. Recently, by using the Minkowski difference, Karaman et al. [6] introduced the following partial order relations on the family of nonempty bounded sets, namely, they are reflexive, transitive and antisymmetric on $\mathcal{B}^*(Y)$.

Definition 2.1. [6] Let $A, B \in \mathcal{P}_0(Y)$.

$A \leq_C^{m_1} B \Leftrightarrow (B \dot{-} A) \cap C \neq \emptyset$.

$A \prec_C^{m_1} B \Leftrightarrow (B \dot{-} A) \cap \text{int } C \neq \emptyset$.

$A \leq_C^{m_2} B \Leftrightarrow (A \dot{-} B) \cap -C \neq \emptyset$.

$A \prec_C^{m_2} B \Leftrightarrow (A \dot{-} B) \cap -\operatorname{int} C \neq \emptyset$.

In the rest of this paper, we only consider the order relation $\preceq_C^{m_2}$, since we can obtain the corresponding results for the order relation $\preceq_C^{m_1}$.

Let K be a nonempty subset of X. Let $F: K \rightrightarrows Y$ be a set-valued mapping. We consider the following set optimization problem with set order $\preceq_C^{m_2}$ (for short, m_2-SOP):

$$(m_2 - \text{SOP}) \ \min_C F(x) \text{ subject to } x \in K.$$

When the set K and the mapping F are perturbed by a parameter λ which varies over a subset Λ in a normed space, we consider the following parametric set optimization problem with set order $\preceq_C^{m_2}$ (for short, m_2-PSOP):

$$(m_2\text{-PSOP}) \ \min_C F(x, \lambda) \text{ subject to } x \in K(\lambda).$$

Definition 2.2. [6] An element $x_0 \in K$ is said to be

a m_2-minimal solution of (m_2-SOP) if there does not exist any $x \in K$ with $F(x) \neq F(x_0)$ such that $F(x) \preceq_C^{m_2} F(x_0)$, that is, either $F(x) \npreceq_C^{m_2} F(x_0)$ or $F(x) = F(x_0)$ for any $x \in K$;

a weak m_2-minimal solution of (m_2-SOP) if there does not exist any $x \in K$ such that $F(x) \prec_C^{m_2} F(x_0)$.

Let $\operatorname{argmin}_{m_2}(K, F)$ and $\operatorname{argmin}_{wm_2}(K, F)$ denote the m_2-minimal solution set of (m_2-SOP) and the weak m_2-minimal solution set of (m_2-SOP), respectively. Besides, the solution concepts of the problem (m_2-PSOP) can be similarly defined. For each $\lambda \in \Lambda$, let $S(\lambda)$ and $S_w(\lambda)$ denote the m_2-minimal solution set of (m_2-PSOP) and the weak m_2-minimal solution set of (m_2-PSOP), respectively. Throughout the paper, we always assume that $\operatorname{argmin}_{m_2}(K, F) \neq \emptyset$ and $S(\lambda) \neq \emptyset$.

In order to give the relationship between the sets $\operatorname{argmin}_{m_2}(K, F)$ and $\operatorname{argmin}_{wm_2}(K, F)$, we need to recall a vital conclusion presented by Karaman et al. in [6].

Lemma 2.1. If $A \in \mathcal{B}^*(Y)$, then $A \dot{-} A = \{0_Y\}$.

Proposition 2.1. Assume that $F(x)$ is bounded for each $x \in K$. Then

$$\operatorname{argmin}_{m_2}(K, F) \subseteq \operatorname{argmin}_{wm_2}(K, F).$$

Proof. Suppose to the contrary that there exists $x_0 \in \operatorname{argmin}_{m_2}(K, F)$ such that $x_0 \notin \operatorname{argmin}_{wm_2}(K, F)$. Then, there exists $y_0 \in K$ such that

$$\left(F(y_0) \dot{-} F(x_0)\right) \cap (-\operatorname{int} C) \neq \emptyset. \tag{1}$$

This together with Lemma 2.1 gives us $F(x_0) \neq F(y_0)$. Moreover, (1) also implies that $F(y_0) \preceq_C^{m_2} F(x_0)$. Thus, it follows that $x_0 \notin \operatorname{argmin}_{m_2}(K, F)$, which leads to a contradiction.

The following example is given to show that the statement $\operatorname{argmin}_{m_2}(K, F) \subseteq \operatorname{argmin}_{wm_2}(K, F)$ may be not true if the values of F are unbounded.

Example 2.1 Let $K = [0,1], Y = \mathbb{R}^2, C = \mathbb{R}_+^2$ and $e = (1,1)$. Let $F: K \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \mathbb{R}e, & \text{if } x \in \{0,1\}, \\ \{0_{\mathbb{R}^2}\}, & \text{if } x \in (0,1). \end{cases}$$

It is easy to verify that $\operatorname{argmin}_{m_2}(K, F) = \{0,1\}$ and $\operatorname{argmin}_{wm_2}(K, F) = \emptyset$.

Definition 2.3. [21] A topological space T is said to be path connected (or arcwise connected) if for any $x, y \in T$, there exists a continuous mapping $\gamma: [0,1] \to T$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Definition 2.4. [22] Let T_1 and T_2 be two topological vector spaces. A set-valued mapping $G: T_1 \rightrightarrows T_2$ is said to be

lower semicontinuous (l.s.c.) at $\bar{t} \in T_1$ iff, for every open set $V \subseteq T_2$ with $G(\bar{t}) \cap V \neq \emptyset$, there is a neighbourhood $N(\bar{t})$ of $\bar{t}$ such that for any $t \in N(\bar{t})$ with $G(t) \cap V \neq \emptyset$;

upper semicontinuous (u.s.c.) at $\bar{t} \in T_1$ iff, for every open set $V \subseteq T_2$ with $G(\bar{t}) \subseteq V$, there is a neighbourhood $N(\bar{t})$ of $\bar{t}$ such that for any $t \in N(\bar{t})$ with $G(t) \subseteq V$;

Hausdorff upper semicontinuous (H-u.s.c.) at $\bar{t} \in T_1$ iff, for each neighbourhood U of 0_{T_2}, there is a neighbourhood $N(\bar{t})$ of $\bar{t}$ such that for any $t \in N(\bar{t})$ with $G(t) \subseteq G(\bar{t}) + U$.

We say that G is 1.s.c. (resp. u.s.c.) on T_1, if it is 1.s.c. (resp. u.s.c.) at each $t \in T_1$. G is said to be continuous on T_1 if it is both l.s.c. and u.s.c. on T_1.

Proposition 2.2 [23, 24] Assume that T_1 and T_2 be two normed vector spaces. Let $G: T_1 \rightrightarrows T_2$ be a set-valued mapping. Then, the following statements are true.

G is l.s.c. at $\bar{t}$ if and only if for any sequence $\{t_n\} \subset T_1$ with $t_n \to \bar{t}$ and for any $\bar{x} \in G(\bar{t})$, there exists $x_n \in G(t_n)$ such that $x_n \to \bar{x}$.

If G has compact values at $\bar{t}$, then G is u.s.c. at $\bar{t}$ if and only if for any net $\{t_n\} \subset T_1$ with $t_n \to \bar{t}$ and any $x_n \in G(t_n)$, there exist $\bar{x} \in G(\bar{t})$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$.

Definition 2.5. [6] For $e \in \text{int } C$, the function $I_e^{m_2}(\cdot,\cdot): \mathcal{P}_0(Y) \times \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$ is defined by

$$I_e^{m_2}(A, B) := \inf\{t \in \mathbb{R}: A \leq_C^{m_2} te + B\}.$$

Proposition 2.3. ([6])

If $A \in \mathcal{B}^*(Y)$ and $B \in \mathcal{P}_0(Y)$, then $I_e^{m_2}(A, B) > -\infty$;

Let $A, B \in \mathcal{P}_0(Y)$. Then, $I_e^{m_2}(A, B) = +\infty$ if and only if $A \dot{-} B = \emptyset$.

Proposition 2.4. ([6]) If $A, B \in \mathcal{P}_0(Y)$ and $A \dot{-} B$ is compact, then $I_e^{m_2}(A, B) = \min\{t \in \mathbb{R}: A \leq_C^{m_2} te + B\}$.

Proposition 2.5. ([6]) Let $A, B \in \mathcal{P}_0(Y)$ and $r \in \mathbb{R}$. Then, the following statements hold.

$I_e^{m_2}(A, B) < r$ if and only if $A \prec_C^{m_2} re + B$.

Let $A \dot{-} B$ be compact, then $I_e^{m_2}(A, B) \leq r$ if and only if $A \leq_C^{m_2} re + B$.

3. Lower Continuity of the Solution Set Mappings of (m_2-PSOP)

In the section, we discuss the lower semicontinuity of the weak m_2-minimal solution set mapping $S_w(\cdot)$ and the m_2-minimal solution set mapping $S(\cdot)$ to (m_2-PSOP) by using a nonlinear scalarization method.

Let us start by showing the lower semicontinuity of a nonlinear scalarization function based on the function $I_e^{m_2}(\cdot,\cdot)$. Let Λ_1 and Λ_2 be two normed vector spaces. Assume that $A: \Lambda_1 \rightrightarrows Y$ and $B: \Lambda_2 \rightrightarrows Y$ are two set-valued mappings. The function $\omega: \Lambda_1 \times \Lambda_2 \to \mathbb{R} \cup \{\pm\infty\}$ is defined as follows:

$$\omega(\mu, \eta) := I_e^{m_2}(A(\mu), B(\eta)) = \inf\{t \in \mathbb{R}: A(\mu) \leq_C^{m_2} te + B(\eta)\},\ (\mu, \eta) \in \Lambda_1 \times \Lambda_2.$$

In the sequel, we always assume that, for any given $\mu \in \Lambda_1, \eta \in \Lambda_2, A(\mu) \in \mathcal{B}^*(Y), B(\eta) \in \mathcal{P}_0(Y)$, and $A(\mu) \dot{-} B(\eta) \neq \emptyset$. Then, by Proposition 2.3, we have

$$-\infty < \omega(\mu, \eta) < +\infty,\ \forall (\mu, \eta) \in \Lambda_1 \times \Lambda_2.$$

Proposition 3.1. Assume that $A(\cdot)$ and $B(\cdot)$ are continuous with nonempty compact values, then $\omega(\cdot,\cdot)$ is continuous on $\Lambda_1 \times \Lambda_2$.

Proof. Firstly, we prove that $\omega(\cdot,\cdot)$ is lower semicontinuous on $\Lambda_1 \times \Lambda_2$. For any $r \in \mathbb{R}$, let

$$M_l = \{(\mu, \eta) \in \Lambda_1 \times \Lambda_2: \omega(\mu, \eta) \leq r\}.$$

It suffices to show that M_l is a closed subset of $\Lambda_1 \times \Lambda_2$. Let $\{(\mu_n, \eta_n)\} \subseteq M_l$ with $(\mu_n, \eta_n) \to (\mu_0, \eta_0)$. Suppose that $(\mu_0, \eta_0) \notin M_l$. Then $A(\mu_0) \not\leq_C^{m_2} re + B(\eta_0)$ by Proposition 2.5 (ii). This indicates that $(A(\mu_0) \dot{-} B(\eta_0) - re) \cap -C = \emptyset$. According to $A(\mu_0) \dot{-} B(\eta_0) \neq \emptyset$, we have

$$x - re \notin -C,\ \forall x \in A(\mu_0) \dot{-} B(\eta_0). \tag{2}$$

Now we claim that there exists $N_0 \in \mathbb{N}$ such that when $n \geq N_0$

$$x_n - re \notin -C,\ \forall x_n \in A(\mu_n) \dot{-} B(\eta_n). \tag{3}$$

If not, then for any n, there exist $m_n \geq n$ and $\bar{x}_{m_n} \in A(\mu_{m_n}) \dot{-} B(\eta_{m_n})$ such that $\bar{x}_{m_n} - re \in -C$. Without loss of the generality, we assume that there exists $\bar{x}_n$ with

$$\bar{x}_n + B(\eta_n) \subseteq A(\mu_n), \tag{4}$$

such that

$$\bar{x}_n - re \in -C, \ \forall n. \tag{5}$$

Thanks to (4), we have $\bar{x}_n + y_n \in A(\mu_n)$ for any $y_n \in B(\eta_n)$. As $A(\cdot)$ is u.s.c. with compact values at μ_0 and $B(\cdot)$ is u.s.c. with compact values at η_0, we see that the sequences $\{y_n\}$ and $\{\bar{x}_n + y_n\}$ have convergent subsequences by Proposition 2.2 (ii). Consequently, $\{\bar{x}_n\}$ has a convergent subsequence $\{\bar{x}_{n_k}\}$. Without loss of the generality, we assume that $\bar{x}_n \to x_0$. Now, we show that $x_0 \in A(\mu_0) \dot{-} B(\eta_0)$. Indeed, for any $b \in B(\eta_0)$, by the lower semicontinuity of the mapping $B(\cdot)$, there exists $b_n \in B(\eta_n)$ such that $b_n \to b$. It follows from (4) that there exists $a_n \in A(\mu_n)$ such that

$$\bar{x}_n + b_n = a_n. \tag{6}$$

Since $A(\cdot)$ is u.s.c. with compact values at μ_0, by Proposition 2.2 (ii), there exist $a_0 \in A(\mu_0)$ and a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $a_{n_k} \to a_0$. Combining this with $\bar{x}_n \to x_0, b_n \to b$ and (6), we arrive at $x_0 + b = a_0 \in A(\mu_0)$. Moreover, by the arbitrariness of $b \in B(\eta_0)$, we have $x_0 + B(\eta_0) \subseteq A(\mu_0)$, i.e., $x_0 \in A(\mu_0) \dot{-} B(\eta_0)$. With the help of (5), $\bar{x}_n \to x_0$ and taking into account the closedness of C, we have

$$x_0 - re \in -C,$$

which contradicts (2). Hence, (3) holds. This implies that $A(\mu_n)$ and so $\omega(\mu_n, \eta_n) > r$ by Proposition 2.5 (ii). This leads to a contradiction. Therefore, M_l is a closed subset of $\Lambda_1 \times \Lambda_2$ and so the proof is completed.

Define $\varphi: X \times X \times \Lambda \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\varphi(x, y, \lambda) = I_e^{m_2}(F(y, \lambda), F(x, \lambda)), \ \forall (x, y) \in X \times X.$$

Lemma 3.1. For each $\lambda \in \Lambda$, one has

$$S_w(\lambda) = \{x \in K(\lambda): \varphi(x, y, \lambda) \geq 0, \ \forall y \in K(\lambda)\}.$$

Proof. By the definition of m_2-minimal solution set of (m_2-PSOP), for each $\lambda \in \Lambda$, we have $x \in S_w(\lambda)$ if and only if

$$F(y, \lambda) \leq_C^{m_2} F(x, \lambda), \ \forall y \in K(\lambda).$$

This indicates that $\varphi(x, y, \lambda) \geq 0$ for any $y \in K(\lambda)$ by Proposition 2.5 (i). Henceforth, the proof is complete.

Next, we prove the lower semicontinuity of $S_w(\cdot)$ and $S(\cdot)$. Let $\hat{S}: \Lambda \rightrightarrows X$ be defined by

$$\hat{S}(\lambda) := \{x \in K(\lambda): \varphi(x, y, \lambda) > 0, \ \forall y \in K(\lambda)\}.$$

In the sequel, we assume that $\hat{S}(\lambda) \neq \emptyset$ for each $\lambda \in \Lambda$. In order to establish the main results, we need the following several lemmas.

Lemma 3.2. Let $\lambda_0 \in \Lambda$. Assume that

$K(\cdot)$ is continuous with nonempty compact values at λ_0;

$F(\cdot,\cdot)$ is continuous with nonempty compact values on $K(\lambda_0) \times \{\lambda_0\}$;

$F(y, \lambda) \dot{-} F(x, \lambda) \neq \emptyset$ for any $x, y \in K(\lambda_0)$ and for any $\lambda \in \Lambda$.

Then $\hat{S}(\cdot)$ is l.s.c. at λ_0.

Proof. To prove the result by contradiction, suppose that $\hat{S}(\cdot)$ is not l.s.c. at λ_0. Then by Proposition 2.2 (i), there exist a sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0$ and $x_0 \in \hat{S}(\lambda_0)$ such that for any $x_n \in \hat{S}(\lambda_n)$, we have $x_n \nrightarrow x_0$.

From $x_0 \in \hat{S}(\lambda_0)$, we have $x_0 \in K(\lambda_0)$. As $K(\cdot)$ is l.s.c. at λ_0, there exists $\bar{x}_n \in K(\lambda_n)$ such that $\bar{x}_n \to x_0$. By the above contradiction assumption, there exists a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ such that $\bar{x}_{n_k} \notin \hat{S}(\lambda_{n_k})$, for any $k \in \mathbb{N}$. Without loss of the generality, we assume that $\bar{x}_n \notin \hat{S}(\lambda_n)$, for any $n \in \mathbb{N}$. Then there exists $y_n \in K(\lambda_n)$ such that

$$\varphi(\bar{x}_n, y_n, \lambda_n) \leq 0,\ \forall n \in \mathbb{N}. \quad (7)$$

Since $K(\cdot)$ is u.s.c. with nonempty compact values at λ_0, by Proposition 2.2 (ii), there exist $y_0 \in K(\lambda_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0$. It follows from Proposition 3.1 that $\varphi(\cdot,\cdot,\cdot)$ is lower continuous on $K(\lambda_0) \times K(\lambda_0) \times \{\lambda_0\}$. This together with (7) states that

$$\varphi(x_0, y_0, \lambda_0) \leq 0,$$

which contradicts $x_0 \in \hat{S}(\lambda_0)$. Thus $\hat{S}(\cdot)$ is l.s.c. at λ_0.

Lemma 3.3. Let $\lambda_0 \in \Lambda$. Assume that the following conditions hold.

$K(\cdot)$ is continuous with nonempty compact values at λ_0;

$F(\cdot,\cdot)$ is continuous with nonempty compact values on $K(\lambda_0) \times \{\lambda_0\}$;

$F(y, \lambda_0) \dot{-} F(x, \lambda_0) \neq \emptyset$ for any $x, y \in K(\lambda_0)$.

Then, we have

$$\hat{S}(\lambda_0) \subseteq S(\lambda_0) \subseteq S_w(\lambda_0) \subseteq \text{cl } \hat{S}(\lambda_0). \quad (8)$$

Proof. We claim that

$$\hat{S}(\lambda_0) \subseteq S(\lambda_0) \subseteq S_w(\lambda_0). \quad (9)$$

Indeed, for any $x \in \hat{S}(\lambda_0)$, by Proposition 2.5 (ii), we have

$$F(y, \lambda_0) \npreceq_C^{m_2} F(x, \lambda_0),\ \forall y \in K(\lambda_0).$$

This implies that $x \in S(\lambda_0)$ and so $\hat{S}(\lambda_0) \subseteq S(\lambda_0)$. Simultaneously, taking into account $S(\lambda_0) \subseteq S_w(\lambda_0)$, we see that (9) is valid. Besides, the inclusion $S_w(\lambda_0) \subseteq \text{cl } \hat{S}(\lambda_0)$ holds obviously. As a consequence, (9) is true and the proof is completed.

Theorem 3.1. Let $\lambda_0 \in \Lambda$. Assume that the following conditions hold.

$K(\cdot)$ is continuous with nonempty compact values at λ_0;

$F(\cdot,\cdot)$ is continuous with nonempty compact values on $K(\lambda_0) \times \{\lambda_0\}$;

$F(y, \lambda) \dot{-} F(x, \lambda) \neq \emptyset$ for any $x, y \in K(\lambda_0)$ and for any $\lambda \in \Lambda$.

Then, $S(\cdot)$ is l.s.c. at λ_0. Moreover, $S_w(\cdot)$ is l.s.c. at λ_0.

Proof. Since the proof is similar to one for the mapping $S_w(\cdot)$, we only prove that $S(\cdot)$ is l.s.c. at λ_0. Indeed, for any $x \in S(\lambda_0)$ and for any neighborhood $U(x)$ of x, and noting that $S(\lambda_0) \subseteq \text{cl } \hat{S}(\lambda_0)$ obtained in Lemma 3.3, we have

$$U(x) \cap \hat{S}(\lambda_0) \neq \emptyset.$$

By Lemma 3.2, we have $\hat{S}(\cdot)$ is l.s.c. at λ_0. Thus there exists a neighborhood $U(\lambda_0)$ of λ_0 such that

$$\hat{S}(\lambda) \cap U(x) \neq \emptyset,\ \forall \lambda \in U(\lambda_0).$$

Since $\hat{S}(\lambda) \subset S(\lambda)$ for each $\lambda \in \Lambda$, we have

$$S(\lambda) \cap U(x) \neq \emptyset,\ \forall \lambda \in U(\lambda_0).$$

This means that $S(\cdot)$ is l.s.c at λ_0.

Remark 3.1 We would like to mention that our main results in this section are different from those in [25-28]. In fact, we study the lower semicontinuity of the minimal solution mapping $S(\cdot)$ and the weak minimal solution mapping $S_w(\cdot)$ for parametric set optimization problems with set order relation $\leq_C^{m_2}$, while [25-28] discuss lowersemicontinuity of the minimal solution mapping $S(\cdot)$ and the weak minimal solution mapping $S_w(\cdot)$ for parametric set optimization problems involving the lower set less relation $\leq^l$ or upper set less relation $\leq^u$. Besides, in this section, we used the nonlinear scalarization method to establish the density result and then give the sufficient conditions for the semicontinuity of the minimal solution mapping $S(\cdot)$ and the weak minimal solution mapping $S_w(\cdot)$ to (m_2-PSOP). The method does not need the any convexity of the objective function and so it is different from level set mappings or monotonicity approaches proposed in [25-28].

Remark 3.2 Recently, Preechasilp and Wangkeeree [29] obtained the lower semicontinuity of the m_1-minimal solution mapping under the assumption that the objective mapping F has converse m_1-property. It is worth noting that the property can lead to the following conclusion when $x_n \to x_0, y_n \to y_0$ and $\lambda_n \to \lambda_0$:

$$F(y_0,\lambda_0) \leq_C^{m_1} F(x_0,\lambda_0) \Rightarrow F\left(y_{n_0},\lambda_{n_0}\right) \leq_C^{m_1} F\left(x_{n_0},\lambda_{n_0}\right) \text{ for some } n_0 \in \mathbb{N}.$$

The may be not reasonable from the point of view of locally sign-preserving property of limit.

Now, we give an example to illustrate that Theorem 3.1 is applicable, but Theorem 3.6 in [29] is not applicable.

Example 3.1 Let $X = \mathbb{R}, Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Let $\Lambda = [0,1]$ and $K: \Lambda \rightrightarrows X$ be defined by $K(\lambda) = \{x \in \mathbb{R}: [-\lambda, \lambda]\}$ for each $\lambda \in \Lambda$. Let $F: X \times \Lambda \rightrightarrows Y$ be defined as follows:

$$F(x,\lambda) = (x+\lambda, \sin\, x) + \mathbb{B}_Y.$$

It is not hard to check that the assumptions (i)-(iii) are satisfied in Theorem 3.1. Now, we check that the assumption (iv) holds. Indeed, for any $\lambda \in \Lambda$, for any $x_1, x_2 \in K(\lambda)$ (without loss of generality, we assume that $x_1 \leq x_2$) and for any $\alpha \in [0,1]$, one has

$$F(\alpha x_1 + (1-\alpha)x_2, \lambda) \leq_C^{m_2} F(x_2,\lambda).$$

Hence, $S(\cdot)$ is 1.s.c. on Λ by Theorem 3.1. However, F does not have converse m_1-property. Indeed, let $x_0 = y_0 = \lambda_0 = 0$. Obviously, $F(y_0,\lambda_0) \leq_C^{m_1} F(x_0,\lambda_0)$ and there exist sequences $\{x_n\} = \left\{\frac{1}{2n}\right\}, \{y_n\} = \left\{\frac{1}{n}\right\}$ and $\{\lambda_n\} = \left\{\frac{1}{n}\right\}$ such that for all $n \in N$ with $F(y_n,\lambda_n) \not\leq_C^{m_1} F(x_n,\lambda_n)$. Hence F does not have the converse m_1-property and so Theorem 3.6 in [29] is not applicable in this example.

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