ISSN: 2664-9640

Comparison of Three Approaches to Solve the 1-D FEM

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Abstract

In this paper, to compare the accuracy difference of different weighted residual methods in solving boundary value problems, we choose the point-matching method, the subdomain collocation method and the Galerkin method for solving the boundary value problem of second-order differential equation. The computational domain of differential equation is divided into those different number of subdomains. With the decreasing of the number of subdomains, the accuracy changes of the result function are compared by the three different methods. When decreasing the number of subdomain partition, the variation for the result function from Galerkin method is obviously smaller than the other two methods.

Keywords

Galerkin method, point-matching method, subdomain collocation method.

1. Introduction

Generally speaking, there are exact solutions and numerical solutions in solving differential equations. For complex problems, it is difficult to solve the exact solutions, so the approximate solutions can meet the expected requirements. The weighted residual method is a semi-analytical method, which originated in the 1960s and is a numerical method for solving approximate solutions of differential equations. Solution of weighted residual method does not depend on the existence of functional and has universality.

Different weighted residual methods can be constituted by choosing different weight function constitute. At present, the main weighted residual methods are the point-matching method, the subdomain collocation method and the Galerkin method. A. C. Cangellari proposed the finite element method in time domain based on the point-matching method and directly calculated the Maxwell equation [1]. X. Yuan proposed a subdomain collocation method for solving multilayer piezoelectric problems, which effectively solved the problem caused by point-matching [2]. P. Wang solve the electromagnetic problem by proposing a galerkin time domain method based on vector wave equation and Maxwell equation [3].

To compare the advantages and disadvantages of the three methods in solving one-dimensional finite element problems, we select a second-order differential equation, divide the computational domain into different parts, and use point-matching method, subdomain collocation method and Galerkin method for solving the boundary value problem

2. Formulations

2.1. Method of Weighted method

The weighted residual method is the basic idea of using approximation method for solving operator equation. Assuming that $u \in H$ and $L(u) = g \in H$, the residual can be defined as

$$T(u) = L(u) - g \in H \tag{1}$$

To obtain the exact solution of the equation, we need to eliminate the residual, that is, the following equation need to be satisfied:

ISSN: 2664-9640

$$\langle T(u), u_i \rangle = 0,\ i = 1,2,\ldots,n \tag{2}$$

In the actual calculation, Let $\widehat{U} = Sp\{u_1, u_2, \ldots, u_n\}$ be a finite-dimensional subspace of function space H and assuming that the solution u can be approximated by $\hat{u} \in H$. The expression of $\hat{u}$ is shown below:

$$\hat{u} = \sum_{j=1}^{n} \beta_j u_j \tag{3}$$

Since $\hat{u}$ is an approximation of u, u can be expressed as

$$u = \hat{u} + e \tag{4}$$

Function space H can be viewed as direct sum of $\widehat{U}$ and its orthocomplement. For getting an accurate approximation, the following equation needs to be satisfied

$$\langle T(\hat{u}), u_i \rangle = 0,\ i = 1,2,\ldots,n \tag{5}$$

Incorporating equation (3) into (1) and Using linear properties of linear operators, we have

$$T(\hat{u}) = \sum_{j=1}^{n} \beta_j L(u_j) - g \tag{6}$$

Incorporating equation (6) into (5), we get the expression of Method of Weighted method

$$\sum_{j=1}^{n} \beta_j \langle L(u_j), u_i \rangle = \langle g, u_i \rangle, i = 1,2,\ldots,n \tag{7}$$

2.2. Point-matching Method, subdomain collocation method and Galerkin Method

In solving specific problems of differential equations, equation (2) can be described as

$$\langle T(u), w \rangle = \int (L(u) - g)\, w d\Omega = 0 \tag{8}$$

Different weighted residual methods can be obtained by choosing different weight functions w. In point-matching method, we chose Dirac impulse function as weighting function, that is, *Lu=f* is valid at the selected sampling point in the computational domain.

$$W_i = \delta(x - x_i) \tag{9}$$

Incorporating equation (9) into (8), we have

$$\langle T(u(x)), \delta(x - x_i) \rangle = T(u(x_i)) \tag{10}$$

Incorporating equation (1) into (10), we get the expression of point-matching method

$$\langle T(u(x)), \delta(x - x_i) \rangle = L(u(x_i) - g(x_i)) = 0 \tag{11}$$

In subdomain collocation method, the weight function is nonzero in given subdomains and zero elsewhere. We can choose the constant, linear function or the sine function as the weight function. The expression of subdomain collocation method can be described as

$$\langle T(u(x)), w_i(x) \rangle = \int L(u(x) - g(x))\, d\Omega = 0 \tag{12}$$

In Galerkin Method, the weight function and the required solution function are in the same function space, that is, . The expression of Galerkin method can be described as

$$\langle T(u(x)), w_i(x) \rangle = \int L((u_j(x))(u_i(x) - g(x)u_i(x))\, d\Omega = 0 \tag{13}$$

3. Solutions

To compare the advantages and disadvantages of three methods in solving one-dimensional finite element problems, we consider the following boundary value problem.

$$\frac{d^2u(x)}{dx^2} = g(x) \tag{14}$$

with the boundary conditions: $u(2) = u(3) = 0$. where *f (x)* is a rectangular functions, shown below

$$g(x) = 1, 2.2 \leq x \leq 2.5 \tag{15}$$

ISSN: 2664-9640

The unknown function is solved by the point-matching method, subdomain collocation method and Galerkin method.

We divide the computational domain of differential equation into n subdomains and select the midpoint of each subdomain as the sampling point. Using point-matching method to calculate. Take n is 7, 15, and 20, and plot the u(x) for different values of n. The image is shown below

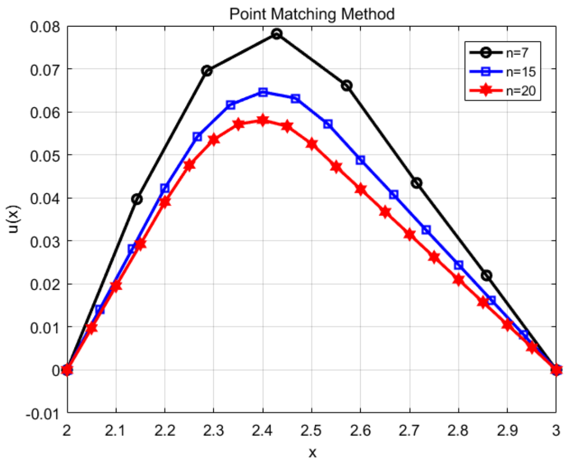

Fig.1 Solution of point-matching method

In subdomain collocation method, We select the impulse function as the weight function and divide the computational domain of differential equation in to n subdomains. Take n is 7, 15, and 20, and plot the u(x) for different values of n. The image is shown below

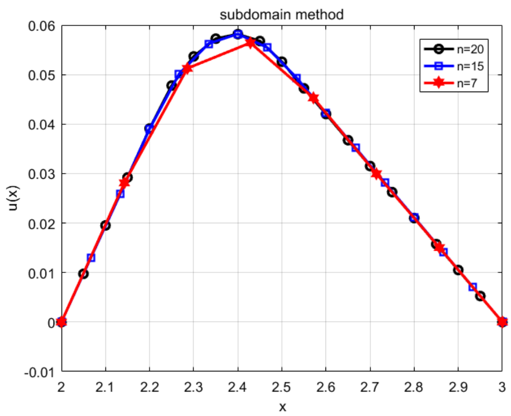

Fig.2 Solution of subdomain collocation method

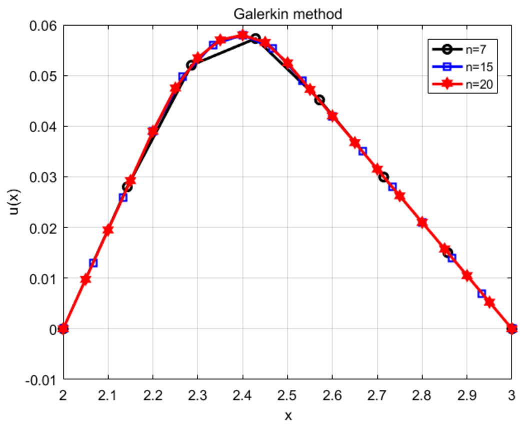

Fig.3 Solution of Galerkin method

In Galerkin method. The weight function and the desired function come from the same function space. In this case, the weight function is defined on the whole computational domain, so the computational domain cannot be divided. Take n is 7, 15, and 20, and plot the u(x) for different values of n. The image is shown below.

Compared with the above four figures, the Galerkin method is superior to the other two methods with the decreasing of n. The point-matching method is the easiest to implement. However, it is easy to lose its precision when the excitation function is discontinuous

References

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