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The uniqueness of the Hofer metric on some Poisson manifolds

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Abstract

This paper studies the uniqueness problem of the Hofer type metric on some special closed Poisson manifolds. Using the leaf structure and the comparison of the norms on the Hamiltonain functions, the relation between the Hofer type norm and the induced norm is studied. If the norm on the Hamiltonian functions satisfies more continuous conditions, then the uniqueness of the Hofer type metric is given.

Keywords

Hofer metric, invariant metric, Poisson diffeomorphism.

1. Introduction

This paper is devoted to discussing the uniqueness of the Hofer metric on the Hamiltonian diffeomorphisms in the Poisson case. As we all know, the group of Hamiltonian diffeomorphisms plays a very important role in the studying of symplectic geometry. The first bi-invariant Finsler metric was defined by Hofer by minimax methods for the abstract Hamiltonian functional[2,3,4]. Viterbo, Polterovich, Lalonde, McDuff and other mathematicians studied this type metric and generalize it to more general case[10,12,13]. In the recent years, Hofer metric was considered by more and more people,the intrinsic properties and geometry of this metric was studied. In the construction of such metrics, the non-degeneracy is not easy to prove, this makes some difficult to extend and compute this famous metric to more general cases. Using Hodge decomposition method, Banyaga found a similar metric on the symplectic diffeomorphisms on a class of compact symplectic manifold[6]. On the identity component of symplectomorphisms, Han defined a metirc and gave some non-extention conditions of the bi-invariant metric[8]. Modifying the norms of the Hamiltonian functions, Lu and Sun constructed a class of bi-invariant metrics on the groups of Hamiltonian diffeomorphisms, and showed the relations of the metric and the continuous metric[16]. Banyaga and Donato studied the lengths of contact isotopies and construced a bi-invariant distance on the strictly contact diffeomorphisms[7], they show that the Hofer mertic can be extended to the symplectic diffeomorphisms under some conditions. Oh and Müller studied the C^0 Hamiltonian Homeomorphisms and defined a Hofer type metric[9].

Ostrover and Wagner studied the extremality of the Hofer metric and proved that if the norms on the Hamiltonian fucntions is less than the L-infinity norm and invariant under the Hamiltonian actions, then the norm is invariant under the volume preserving diffeomorphisms [17]. They discussed the following problem:

Question 1: What kind of norms generate bi-invariant metric on the Hamiltonian diffeomorphisms?

They even proved that if the two norms are not equivalent, then the associated pseudo-distance function on the Hamiltonian diffeomorphisms is zero. Buhovsky and Ostrover studied the uniqueness of the Hofer's geometry, they proved that if a bi-invariant metric generated by a norm satisfies that the norm is continuous in the smooth topology, then the norm generates the same topology with the topology generated by Hofer norm[18].

In this paper we want to study similar questions on the Poisson manifold. The Poisson structure is different with the symplectic structure, it has difficulties to define Hofer type metric directly by Hofer's methods. The Hofer type metric was concerned by Sun and Zhang[14], the authors defined a Hofer type metric on a class of special Poisson manifold with more conditions on the manifold. T. Rybicki found a genuine metric on the Hamiltonian group when all proper leaves of the corresponding symplectic foliation is dense[15].

Untill now, there are little results about the uniqueness about the Hofer metric on Poisson manifold. We want to ask whether Ostrover and Buhovsky's result still hold in the Poisson case. We will study the uniqueness on some Poisson manifolds with the help of the geometry structure. For the convenience of discussion, we assume the Poisson manifold is closed and all the symplectic leaves are also closed, we will named it type G Poisson manifold. The Hamiltonian fuctions are all well defined. The main theorem of this paper is the following:

Theorem 2. Let $(M,\{\ \})$ be a Poisson manifold of type G, $\|\ \|$ be a Hamiltonian invariant norm on the Hamiltonian functions. The norm $\|\ \|$ satisfies the following inequality:

$$\|\ \| \le \mathrm{C}\,\|\ \|_{p,\infty} \tag{1}$$

Here C is a constant. If the two norms are not equivalent, then the associated pseudo distance function is zero.

2. Preliminaries

In this section, we will introduce some basic definitions and notations in Poisson geometry, detailed facts can be found in [1,4,5,11].

Definition 3[5]. A Poisson bracket on a manifold is a bilinear operation $\{\ \}$ on the smooth functions of the manifold M. For the smooth functions f, g, h on the manifold, the following hold:

$(M,\{\ \})$ is a Lie algebra.

$\{\ \}$ is a derivation in each factor, that is

$$\{f,gh\} = g\{f,\mathrm{h}\} + h\{f,g\} \tag{2}$$

For a smooth function H, the Poisson vector is defined by

$$X_H[\mathrm{g}] = \{g,H\} \tag{3}$$

For all smooth function g.

Similarly with the symplectic diffeomorphism, if a diffeomorphism keeps the Poisson bracket, we call it Poisson diffeomorphism. Since the manifold is closed, the flow of the Poisson vector always exists, the Hamiltonian diffeomorphism is the time one map of the Hamiltonian flow. The set of all Hamiltonian diffeomorphism is called Hamiltonian diffeomorphism group.

Next we recall the construction of the Hofer metric, for $H(t,x) \in C^\infty([0,1]\times M,\mathbb{R})$, the Linfinity-norm is defined as

$$\|H_t\|_\infty = \int_0^1 \max_x H(t,x) - \min_x H(t,x)dt \tag{4}$$

Definition 4[4]. Let φ be a Hamiltonian diffeomorphism on the standard symplectic space, the Hofer norm is

$$\|\varphi\| = \inf\{\|H_t\| \,|\, \phi_H^1 = \varphi\} \tag{5}$$

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In the Poisson case, the Hamiltonian function and the Hamiltonian flow are not one-one correspondence, a Hamiltonian flow may have many Hamiltonian functions, we introduce the following norms involving the Casimir functions.

For smooth function $f \in C^{\infty}(M)$, and Casimir function $f_2 \in C^{\infty}(M)$, we define the Poisson Linfinity norm and Poisson type Hofer norm as following:

$$\| f \|_{p,\infty} = \inf \{ \| f_1 \|_{\infty} \mid f = f_1 + f_2, f_2 \in Cas(M) \} \tag{6}$$

$$\|\varphi\|_{p} = \inf\{ \|H_t\|_{p,\infty} \mid \phi_H^1 = \varphi \} \tag{7}$$

3. Proof of main results

In this part we will give the proof of the main results. We first analyse the assumptions of the Poisson manifod. Poisson manifold can be seen the union of symplectic leaves, each leaves can be viewed as a symplectic manifold. The restriction of the Poisson structure on the symplectic leaf is just the symplectic sructure. So we can get the relation of the Poisson manifold and the symplectic leaves, the symplectic methods can be used for the proof. Since the manifold is type G, we can get that

Lemma 5. The Hamiltonian function is well defined on each symplectic leaves for the G type Poisson manifold.

So the the Hofer type metric can be well defined. Now we consider the estimate of the norms of the Hamiltonian function.

Lemma 6[17]. For smooth Hamiltonian functions on the closed symplectic manifold, if $Sup\{\| f_n \|_{\infty}\} \prec \infty$, and vol(support(f_n)) $\to 0$, then the following holds:

$$\| f_n \| \to 0 \tag{8}$$

From the assumption and the construction of the norm, we know that:

$$\text{vol}(\text{support}(f_n|_L)) \to 0 \tag{9}$$

$$Sup\{\| f_n|_L \|_{\infty}\} \prec \infty \tag{10}$$

(9) and (10) hold on each symplectic leaf, so we have the following lemma:

Lemma 7. For smooth Hamiltonian functions on Poisson G manifold, if $Sup\{\| f_n \|_{p,\infty}\} \prec \infty$, and vol(support(f_n)) $\to 0$, then the following hold:

$$\| f_n \| \to 0 \tag{11}$$

Proof of Theorem 2: Now we give the outline of the proof. According to the assumption of G manifold and Lemma 5,6,7, we reduce the problems on each leaf, we know that on each leaf, the norm of the Hamiltonian diffeomorphism is 0. And hence we can get the conclusion by contradiction.

If we give more assumption on the norm of the Hamiltonian functions, we can get the following :

Theorem 3. Let $(M, \{ \ \})$ be a Poisson manifold of type G, $\| \ \|$ be a Hamiltonian invariant norm on the Hamiltonian functions. If the norm is continuous in the C^{∞} topology, then the norm $\| \ \|$ satisfies the following inequality:

$$\| \ \| \leq C \| \ \|_{p,\infty} \tag{12}$$

for some constant C.

Proof: This proof is similar with Theorem2, by the methods and results of [18], we know that this holds on each leaf for each Hamiltonian functions.

Corollary 1. Let $(M,\{\ \})$ be a Poisson manifold of type G, $\|\ \|$ be a Hamiltonian invariant norm on the Hamiltonian functions. If the norm is continuous in the C^∞ topology, then the pseudo norm generated by the norm is either 0 or equivalent to Hofer type metric.
Proof: If the the pseudo norm generated by the norm is not equuivalent to Hofer metric, and is continuous in the C^∞ topology, By Theorem 3, we have

$$\|\ \| \leq C \|\ \|_{p,\infty} \quad (13)$$

By Theorem 2 again, we can get the results.
Remark 1. The assumption of Type G manifold can guarantee the existence of the Hamiltonian functions. The proof reduce the conclusions to the symplectic leaf, we can also try to use the capacity method to prove it, if we do not have the assumption of G manifold, we can try to use the dense assumptions of T. Rybicki, we will discuss this in other article.
Remark 2. The type G Poisson manifold exists, one can consider a ball without a small ball with same center.

4. Conclusion

In this paper we discuss the uniqueness problem of the Hofer type metric on a class of special Poisson G manifold. We study the relation between norms on the Hamiltonian functions and the generating bi-invariant norms on the Hamiltonian diffeomorphisms. Using the symplectic results and the assumptions of the leaf structure, we give several inequalities and show that if the norms on the Hamiltonian satisfy some conditions then the Hofer norm is unique in the bi-invariant sense. This partly answers the extension question from symplectic to Poisson for the uniqueness of the Hofer metric.

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